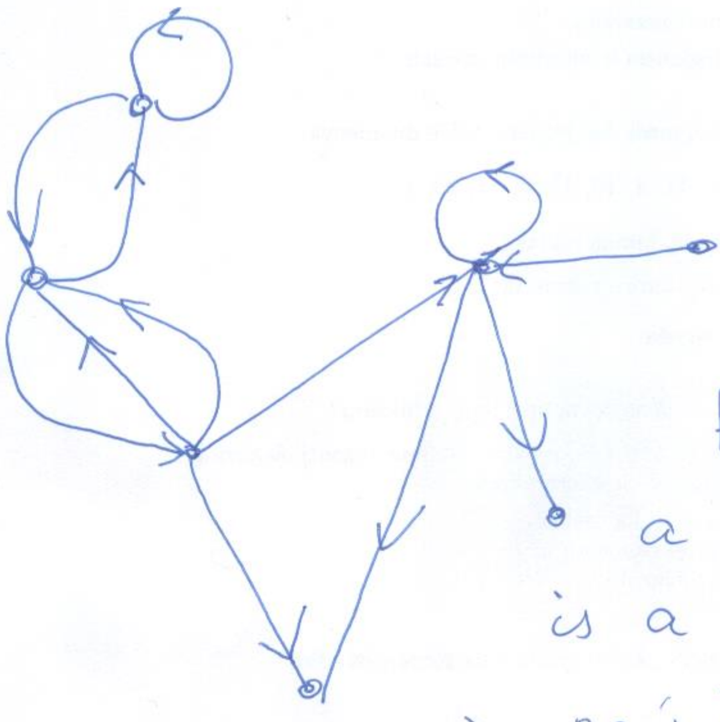
Lecture 4.

Graph theory

Here we start by defining a more general object and later we will use different simplifications of it.



This objet is an example of a <u>multi-graph</u>

Formally, a multi-graph is a collection of

a) points

b) line segments that connect these points.

Definitions. The given points are called vertices of a graph; and directed line segments are called arcs (or lines) of a graph.

Let's denote the set of vertices as

$$V = \{ v_1, v_2, \dots, v_n \}.$$

Then the multi-graph line l can be defined as an element of the following Cartesian product

$$l = (v_i, v_j, k) \in V \times V \times \mathbb{N}.$$

Here v_i is an initial vertex of line l, v_j is a final vertex, $k \in \mathbb{N}$ is line number.

If a graph has no parallel lines, then lines can be not numbered, i. e. $l \in V \times V$.

Definition. Line l whose initial and final vertices coincide, is called a loop:

$$l = (v, v, k), \ v \in V, \ k \in \mathbb{N}.$$

or

$$l = (v, v) \text{ if no parallel lines exist.}$$

Definition A multi-graph that contains no parallel lines is a directed graph (or digraph)
A directed graph without loops is a simple digraph.

As was stated above the set of simple digraph lines is a subset of $V \times V$.

Wow - we met an old friend

This subset is a binary relation L in the set of digraph vertices V.

Next we remind you that

$L \subset V \times V$ is irreflexive

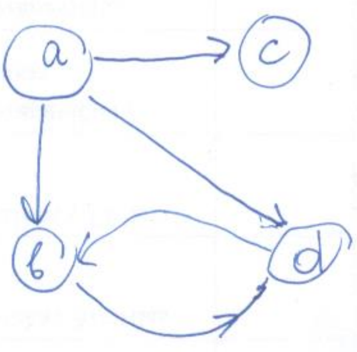
if $(v, v) \notin L$, $\forall v \in V$.

Thus digraph has no loops if the corresponding binary relation is irreflexive.

We define simple digraphs G as a pair of sets V and L:

$G = (V, L)$, $L \subset V \times V$ is any given binary <u>irreflexive</u> relation

Lines of digraph are called arcs (liet. lankas).



$V = \{a, b, c, d\}$

$L = \{(a,b), (a,c), (c,d), (b,d), (d,b)\}$.

Unidirected Graph

Suppose that simple digraph $G = (V, L)$ is a <u>symmetric graph</u>, i.e. L is a symmetric relation

$$(v_i, v_j) \in L \Rightarrow (v_j, v_i) \in L.$$

Such graphs are called

simple undirected graphs. (or

simple graphs).

If a simple undirected graph can have additional loops, then it is called a graph.

For a simple graph we replace any pair of arcs (v_i, v_j), (v_j, v_i) with a pair $\{v_i, v_j\} = e$ and call it an edge.

$$G = (V, E), \quad E = \{e_j\}.$$

$$E = \{\{v_i, v_j\}, \ v_i, v_j \in V\} \subset V^{(2)}$$

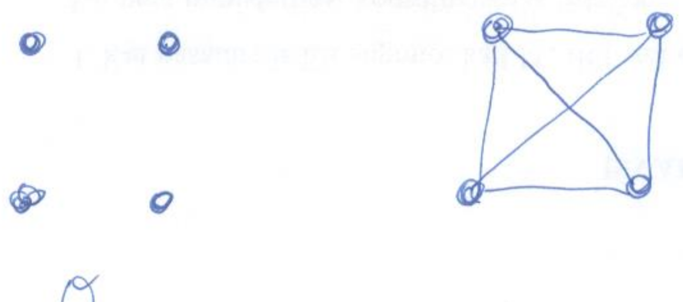
$$V^{(2)} = \{\{v_i, v_j\}, \ i < j, \quad i, j = 1, \dots, n\}$$

1. The order n of graph $G = (V, E)$ is the cardinality (order) of its vertex set $n = |V|$.

2. $O_n = (V, \phi)$ is called an empty graph, $n = |V|$.

3. Graph $G = (\phi, \phi)$ is called a null graph.

4. The graph of order n that contains all available edges, is called a complete graph.



$\frac{n(n-1)}{2}$ edges

(why ??)

O_4

1. Two vertices v_i and v_j of graph $G=(V,E)$ are said to be adjacent vertices if $\{v_i, v_j\} \in V$.

For a digraph $G=(V,L)$ v_i and v_j are adjacent vertices if $(v_i, v_j) \in L$ and $(v_j, v_i) \in L$.

2. Any of vertices v_i and v_j is said to be incident vertex to the edge $\{v_i, v_j\}$.

Two edges are said to be adjacent edges (arcs) if there exists a vertex that is incident to both edges.

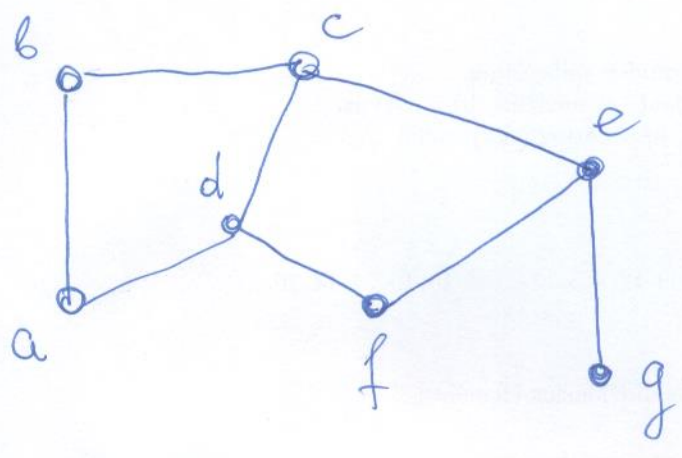
Graph connectivity

Let $G = (V, E)$ be an arbitrary graph.

<u>Definition.</u> A <u>finite</u> sequence of graph vertices and edges

$$W = v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \dots, v_{i_{k-1}}, e_{i_k}, v_{i_k}$$

where $e_{i_j} = \{v_{i_{j-1}}, v_{i_j}\}$ is called a walk.



$$W = a, \{a, b\}, b, \{b, c\}, c, \{c, d\}, d.$$

A simpler notation is given by

$$W_1 = (a, b, c, d).$$

There exist other walks in this graph

$W_2 = (c, e, g)$

(Note, that $\{c, e\}, \{e, g\} \in E$).

<u>Definition</u>. A walk is called a circuit if all its <u>edges</u> are distinct. (grandine)

Vertices v_{i_0} and v_{i_k} are called terminal vertices of a walk and $v_{i_1}, \dots, v_{i_{k-1}}$ are called internal vertices.

v_{i_0} – an initial (start) vertex

v_{i_k} – a final (end) vertex of a walk.

A given circuit is an open circuit if its terminal vertices don't coincide.

Otherwise we have a closed circuit

An open circuit is called a path.

A closed circuit is said to be a cycle.

A circuit is said to be a simple circuit if all its internal vertices are different from each other.

Example: $M = (a, b, c, e, f, d, c, b, d)$

$M_1 = (a, b, c, d, e)$

$M_2 = (c, e, g)$ $M_3 = (b, c, e, f, d, b)$

$M_4 = (a, b, d, c, e, f, d, a)$

Give a classification of these walks

Definition. Graph G is a connected graph if arbitrary pair of its vertices $v_i, v_j \in V$ can be connected with a path.

Otherwise G is said to be a disconnected graph.

In this case its vertices V can be divided into two disjoint blokks V_1 and V_2 : $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \phi$ that $\forall v, w \in V$
$v \in V_1$ and $w \in V_2 \Rightarrow \overline{\exists} \{v, w\} \in E$ (and $\overline{\exists} (w, v) \in L$ for digraphs).

This partition process can be continued if some blokk V_j of vertices is disconnected:

$$V = V_1 \cup V_2 \dots \cup V_m.$$

where $V_i \cap V_j = \phi \quad \forall i, j$

and any pair of vertices from $V_s \subset V$ can be connected with a path in $G = (V, E)$, but any pair of vertices which belong to different subsets V_i and V_j can't.

Definition Let's denote $E_j \subset E$ subset of E, which elements are edges of E that are incident at least to one vertex from blokk V_j

$G_j = (V_j, E_j)$ is said to be a connected component of graph $G=(V,E)$

Example $G = (V, E)$

$$V = \{1, 2, 3, 4, 5, 6, 7\}$$

$$E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}, \{6,7\}\}$$

1) Find all connected components of this graph.
2) How many isolated vertices exist?

Definition Let $G = (V, E)$ be an arbitrary graph. In the set V we define a graph vertex-connectivity relation: $\rho \subset V \times V$

$$(u, w) \in \rho \iff (u = w) \vee \exists (u, v_1, \ldots, v_m, w)$$

i.e. vertex u is related ~~to~~ vertex w iff they can be connected with a path

Definition. The length of a path is equal to a number of edges in a path.

In many applications it is important to find the longest path connecting some vertexes of the given graph.

Theorem. Any two paths of maximal length in any connected graph contain at least one vertex that belongs to both of them.

Proof. Assume that $P_1 = (v_0, v_1, \dots, v_k)$ and $P_2 = (v_0', v_1', \dots, v_k')$ are two paths in $G = (V, E)$, both of maximal length.

In addition we assume, that

$$v_i \neq v_j' \quad \forall\, 0 < i, j \leq k.$$

Since G is a connected graph any pair of its vertices can be connected with some path.

There is a path P_3 connecting some $v_i \in P_1$ to $v_j' \in P_2$ such that P_3 shares no vertices with $P_1 \cup P_2$ other than v_i and v_j'. Note, that there may be no vertices u_l in P_3

$$P_3 = (v_i, u_1, .., u_b, v_j')$$

A proof of this statement is simple. Take any $v_e \in P_1$ and $v_m' \in P_2$. Construct a path

$$P_4 = (v_e, \tilde{u}_1, \tilde{u}_2, .., v_r, \tilde{u}_s, ..,$$

$$\ldots, v_i, \tilde{u}_p, \ldots, v_j', \ldots v_m')$$.

Then the last vertex $v_i \in P_1$ belonging to P_4 and the first vertex $v_j' \in P_2$ define the required bridge P_3.

Let assume that $i \geq \left[\frac{k}{2}\right]$ and $j \leq \left[\frac{k}{2}\right]$. Then we construct a new path

$$P^* = (v_0, \ldots, v_i, u_1, \ldots, u_\ell, v_j', \ldots, v_k').$$

Obviously P^* has length at least $k+1$

This contradicts the assumption that graph G has no paths of length greater than k.

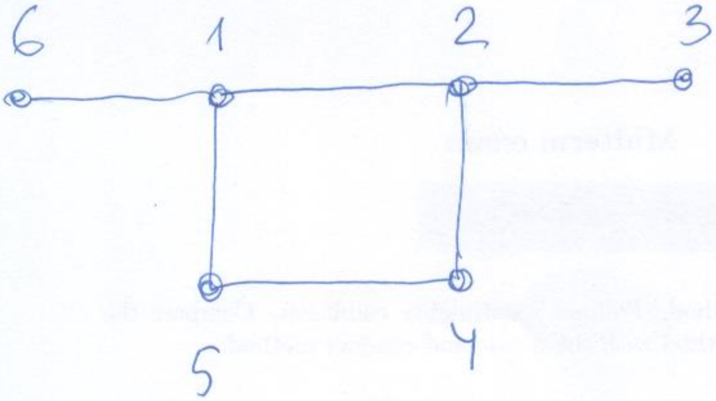
Example.

Graph $G = (V, E)$

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 5\}, \{1, 6\}, \{2, 5\}, \{4, 5\}\}$

Find a path/or paths of maximal length.



Maximal length $|P| = 5$.

$$P_1 = (3, 2, 4, 5, 1, 6)$$

$$P_2 = (6, 1, 2, 4, 5, 1)$$